

## ON THE RATE OF CONVERGENCE OF THE NONLINEAR GALERKIN METHODS

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**ABSTRACT.** In this paper we provide estimates to the rate of convergence of the nonlinear Galerkin approximation method. In particular, and by means of an illustrative example, we show that the nonlinear Galerkin method converges faster than the usual Galerkin method.

### 1. INTRODUCTION

The nonlinear Galerkin methods originate from recent developments in the study of the long-time behavior of dissipative partial differential equations. It is well known that this behavior depends in an essential manner on certain nondimensional bifurcation parameters, such as the Reynolds or the Grashoff numbers for the Navier-Stokes equations (NSE). For small values of these parameters, the solution converges as  $t \rightarrow +\infty$  towards rest or towards a stationary solution  $u_s$ . However, for large values of these parameters, the dynamics of the system, in general, becomes nontrivial. In particular, even if the driving forces are time-independent, the system may display a time-dependent behavior. The solutions converge to a set, the global (universal) attractor, which is compact and invariant under the flow of solutions. This set is the mathematical object describing the permanent turbulent or chaotic regime. An important question, from both dynamical and numerical points of view, is then how to approximate the global attractor. As a partial answer the concepts of inertial manifolds (IM) and approximate inertial manifold (AIM) have been introduced. Let us write the partial differential equation as an abstract differential equation operating in some Hilbert space  $H$ :

$$(1.1) \quad \frac{du}{dt} + \nu Au + R(u) = 0.$$

We consider a basis of  $H$  consisting of eigenvectors of the basic linear dissipative operator  $A$ . We denote by  $P_m$  the orthogonal projection of  $H$  onto the space spanned by the first  $m$  eigenvectors of the operator  $A$ . Also, we set  $Q_m = I - P_m$ . By applying  $P_m$  and  $Q_m$  to (1.1), one obtains the equivalent

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system

$$\begin{aligned} \frac{dp}{dt} + \nu Ap + P_m R(p + q) &= 0, \\ \frac{dq}{dt} + \nu Aq + Q_m R(p + q) &= 0. \end{aligned}$$

An inertial manifold (see Foias, Sell, and Temam [17]) is a finite-dimensional manifold,  $\mathcal{M}$ , which is positively invariant under the flow and attracts all the solutions at an exponential rate. It is sought as the graph of a suitable function  $\Phi: P_m H \rightarrow Q_m H$ . If such a manifold exists, it contains the global attractor, and the dynamics on  $\mathcal{M}$  is completely described by the system of ordinary differential equations

$$\frac{dp}{dt} + \nu Ap + P_m R(p + \Phi(p)) = 0,$$

which is called the associated inertial form (see Constantin, Foias, Nicolaenko, and Temam [6], Foias, Nicolaenko, Sell, and Temam [15], Foias, Sell, and Temam [17], Foias, Sell, and Titi [18], Mallet-Paret and Sell [38], and the references therein).

Unfortunately, the existence of IM is not known for many dissipative partial differential equations, including the 2-D Navier-Stokes equations. However, note that in the latter case, the existence of an inertial form has been derived recently in the 2-D case with periodic boundary condition, see Kwak [35]. Also, except in very special cases (Bloch and Titi [1]), in general, one will not be able to find the explicit form of the inertial manifolds, even if they exist. A substitute, and perhaps computationally more convenient, concept is that of approximate inertial manifold (Foias, Manley, and Temam [13, 14], Foias, Sell, and Titi [18]). An AIM of order  $\varepsilon > 0$  is a finite-dimensional manifold  $\widetilde{\mathcal{M}}$  such that the solutions enter in finite time an  $\varepsilon$ -neighborhood of  $\widetilde{\mathcal{M}}$ . In particular, this neighborhood contains the global attractor. Therefore,  $\widetilde{\mathcal{M}}$  provides an approximation of order  $\varepsilon$  of the global attractor. The AIM are obtained as graphs of functions  $\Phi_{\text{app}}: P_m H \rightarrow Q_m H$ ; the associated approximate inertial form is then the ODE

$$(1.2) \quad \frac{dp}{dt} + \nu Ap + P_m R(p + \Phi_{\text{app}}(p)) = 0.$$

The simplest AIM turns out to be the linear space  $P_m H$  for which  $\Phi_{\text{app}}(p) \equiv 0$ . Nonlinear AIM providing a better-order approximation than  $P_m H$  have been considered by several authors: Foias et al. [13, 12, 14, 18], Marion [39, 40], Titi [55, 56], Jolly, Kevrekidis, and Titi [32, 33], Temam [50], and Debussche and Marion [7]. In particular, in the last two references, a method for constructing sequences  $\mathcal{M}_j$  of AIM providing better and better orders of approximation is presented.

The nonlinear Galerkin methods consist in introducing approximate solutions lying on the AIM. One looks for approximate solutions of the form  $u_m = y_m + z_m$ , where  $z_m = \Phi_{\text{app}}(y_m)$ , or  $z_m$  is a finite-dimensional approximation of  $\Phi_{\text{app}}(y_m)$ ; here,  $y_m$  is given by the resolution of the approximate inertial form

$$\frac{dy_m}{dt} + \nu Ay_m + P_m R(y_m + z_m) = 0.$$

Convergence results for such schemes have been obtained by Marion and Temam [41], Jolly et al. [33], and Devulder and Marion [8] for spectral bases. More general bases (finite elements, finite differences) are considered in Marion and Temam [42], Temam [52], and Chen and Temam [4] (in this regard, see also Foias and Titi [23]).

The improvements of the nonlinear Galerkin method over the usual Galerkin method are evidenced by numerical computations that show improved stability and accuracy, and a significant gain in computing time; see Brown, Jolly, Kevrekidis, and Titi [3], Foias, Jolly, Kevrekidis, Sell, and Titi [12], Jauberteau, Rosier, and Temam [31], Jolly et al. [32, 33], Dubois, Jauberteau, and Temam [9], and Dubois, Jauberteau, Marion, and Temam [10]. Also an improved stability condition for these schemes is obtained in Jauberteau et al. [31]. For other computational aspects of the nonlinear Galerkin method, see Graham, Steen, and Titi [25]. Also, see Foias, Jolly, Kevrekidis, and Titi [24], Shen [46], and Temam [52] for other stability issues.

In this paper we are interested in deriving error estimates for the nonlinear Galerkin methods. We consider the 2-D Navier-Stokes equations. We start in §2 by recalling basic results on the functional setting of these equations. Section 3 contains our main results. We consider an abstract AIM,  $\mathcal{M} = \text{graph } \Phi_{\text{app}}$  of order  $\varepsilon$  and the corresponding nonlinear Galerkin method. We obtain in §3.1 error bounds in various norms for this scheme in terms of  $\varepsilon$ . In these estimates, the influence of the spatial discretization is clearly evidenced. In §3.2 we remark that if one assumes, as in Heywood [28], and Heywood and Rannacher [29], that the exact solution is exponentially stable—which usually is difficult to check—then one can obtain error estimates that are uniform in time. We will report the details of this result in a subsequent work. In §4, our results are applied to various AIM that have been constructed. Some corresponding technical proofs are given in the Appendix. Our results justify rigorously the improvements of the nonlinear Galerkin methods with respect to the Galerkin method. For the latter method, in the spectral case that we consider here, the error is of the order of  $(\log \lambda_m)/\lambda_{m+1}$  ( $\lambda_m = m$ th eigenvalue of the Stokes operator), while it is of the order of  $(\log \lambda_m)^\alpha/\lambda_{m+1}^\alpha$ ,  $\alpha > 1$ , for nonlinear Galerkin methods. The value of  $\alpha$  depends on the order of the AIM used in the corresponding approximate inertial form (1.2).

## 2. THE 2-D NAVIER-STOKES EQUATIONS AND DYNAMICAL SYSTEMS

We consider the 2-D Navier-Stokes equations in a bounded domain  $\Omega \subset \mathbf{R}^2$  with the appropriate boundary conditions, which we will discuss later:

$$(2.1) \quad \begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f, \\ \nabla \cdot u &= 0, \end{aligned}$$

where the kinematic viscosity  $\nu > 0$  and the body forces  $f$  are given. The unknowns are the velocity  $u$  and the pressure  $p$ .

It is well known that the Navier-Stokes equations with the appropriate boundary conditions are equivalent to a functional differential equation

$$(2.2) \quad \frac{du}{dt} + \nu Au + Cu + B(u, u) = f,$$

in a certain Hilbert space  $H$ , for instance see Lions [37], Temam [47, 48], and Constantin and Foias [5]. Equation (2.2) is similar to (1.1) with

$$R(u) = Cu + B(u, u) - f.$$

We denote by  $(\cdot, \cdot)$  the inner product in  $H$  and by  $|\cdot|$  the corresponding norm. Here,  $A$  is the Stokes operator, which is a selfadjoint positive operator, with domain  $D(A)$  dense in  $H$ . Moreover,  $A^{-1}$  is compact, and as a result the space  $H$  possesses an orthonormal basis  $w_j$  of eigenfunctions of the operator  $A$ :  $Aw_j = \lambda_j w_j$ ,  $j = 1, 2, \dots$ , where  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . More precisely, there are positive constants  $c_1, c_2$  such that

$$(2.3) \quad c_1 k \leq \frac{\lambda_k}{\lambda_1} \leq c_2 k \quad \text{for } k = 1, 2, \dots$$

(see Métivier [43]). We denote  $V = D(A^{1/2})$  and the corresponding norm in  $V$  by  $\|\cdot\| = |A^{1/2}\cdot|$ . The linear operator  $C$  depends on the boundary conditions. If equations (2.1) are supplemented with periodic or homogeneous (nonslip) boundary conditions, then  $C \equiv 0$ . Otherwise, in the case of nonhomogeneous boundary conditions, the operator  $C$  satisfies

$$(2.4) \quad (\nu Au + Cu, u) \geq \alpha \|u\|^2$$

for some  $\alpha > 0$ , where  $\alpha$  depends on  $\nu$  and the boundary condition (see Hopf [30], also see Lions [37]). Moreover, there exists a positive constant  $K$  such that

$$(2.5) \quad |Cu| \leq K \|u\| \quad \text{for all } u \in V.$$

Hereafter, the constants  $c_1, c_2, c_3, \dots$  are positive constants which are dimensionless, while  $K$  will denote a generic constant which might depend on the physical parameters of the equation. The bilinear operator  $B(u, v)$  enjoys the following properties:

$$(2.6) \quad (B(u, v), w) = -(B(u, w), v) \quad \text{for all } u, v, w \in V,$$

$$(2.7) \quad |(B(u, v), w)| \leq c_3 |u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2} \|w\|^{1/2} \quad \text{for all } u, v, w \in V,$$

$$(2.8) \quad |(B(u, v), w)| \leq c_4 \|u\|_\infty \|v\| |w|$$

for all  $u \in D(A)$ ,  $v \in V$ , and  $w \in H$ ,

$$(2.9) \quad |(B(u, v), w)| \leq c_4 |u| \|v\| \|w\|_\infty$$

for all  $u \in H$ ,  $v \in V$ , and  $w \in D(A)$ ,

where  $\|\cdot\|_\infty$  denotes the  $L^\infty(\Omega)$  norm. Applying the Brezis-Gallouet [2] inequality, one finds that (2.8) and (2.9) become

$$(2.10) \quad |(B(u, v), w)| \leq c_5 \|u\| \|v\| |w| \left( 1 + \log \frac{|Au|^2}{\|u\|^2 \lambda_1} \right)^{1/2}$$

for all  $u \in D(A)$ ,  $v \in V$ , and  $w \in H$ ,

$$(2.11) \quad |(B(u, v), w)| \leq c_5 |u| \|v\| \|w\| \left( 1 + \log \frac{|Aw|^2}{\|w\|^2 \lambda_1} \right)^{1/2}$$

for all  $u \in H$ ,  $v \in V$ , and  $w \in D(A)$ .

We will also need the following inequality borrowed from Titi [53, 54]:

$$(2.12) \quad |(B(u, v), Aw)| \leq c_6 \|u\| \|v\| |Aw| \left( 1 + \log \frac{|Av|^2}{\|v\|^2 \lambda_1} \right)^{1/2}$$

for all  $u \in V, v \in D(A),$  and  $w \in D(A).$

We recall that for  $u_0$  given in  $H,$  the initial value problem (2.1), with the initial condition  $u(0) = u_0,$  possesses a unique solution defined for all  $t > 0$  and such that

$$u \in \mathcal{C}(\mathbf{R}_+; H_{\text{weak}}) \cap L^2_{\text{loc}}(0, \infty; V).$$

If, moreover,  $u_0 \in V,$  then

$$u \in \mathcal{C}(\mathbf{R}_+; V) \cap L^2_{\text{loc}}(0, \infty; D(A)).$$

For further results concerning the existence, uniqueness, and regularity of solution to the NSE, we refer to Constantin and Foias [5], Foias and Temam [22], Henshaw, Kreiss, and Reyna [26], Heywood [27], Ladyzhenskaya [36], Lions [37], Temam [47, 48], and references therein.

*Remark 2.1.* Other dissipative equations can be treated as (2.2), for instance the 2-D Bénard convection problem, 2-D and 3-D convection in porous media (see Graham et al. [25]), 2-D Magneto-hydrodynamics equations, the Kuramoto-Sivaskinsky equation, etc.

### 3. ERROR ANALYSIS FOR THE NONLINEAR GALERKIN METHODS

In this section, we consider an AIM of order  $\varepsilon > 0$  and derive error estimates for the corresponding nonlinear Galerkin method. Section 3.1 deals with the approximation of a general solution of the Navier-Stokes equations, while in §3.2 we consider the case of a stable solution. Applications of our results to various specific AIMs can be found in §4.

**3.1. The general case.** As mentioned in the introduction, we introduce the orthogonal projection  $P_m$  on the space spanned by the first  $m$  eigenfunctions of the Stokes operator  $A,$  and let  $Q_m = I - P_m.$

Let  $\Phi_{\text{app}}$  be a function from  $P_m H (= P_m V)$  (or from  $P_m B_V(0, M_1),$  where  $B_V(0, M_1)$  denotes the ball in  $V$  centered at the origin and of radius  $M_1$ ) into  $Q_m V$  which is supposed to be Lipschitz continuous,

$$(3.1) \quad \|\Phi_{\text{app}}(p_1) - \Phi_{\text{app}}(p_2)\| \leq l \|p_1 - p_2\|,$$

for all  $p_1, p_2$  in  $P_m V$  (or in  $P_m B_V(0, M_1).$

We assume that  $\mathcal{M} = \text{graph}(\Phi_{\text{app}})$  is an AIM of order  $\varepsilon > 0.$  More precisely, we suppose that, for any solution  $u(t)$  of (2.2) such that

$$(3.2) \quad \|u(t)\| \leq M_1 \quad \text{for all } t \geq 0,$$

we have that

$$(3.3) \quad \|Q_m u(t) - \Phi_{\text{app}}(P_m u(t))\| \leq \varepsilon \quad \text{for all } t \geq 0.$$

In the applications,  $B_V(0, M_1)$  is an absorbing ball for (2.2). Then, clearly, (3.3) implies that  $\mathcal{M}$  is an AIM of order  $\varepsilon.$  Indeed, every solution enters, in finite time (for  $t \geq T_* = T_*(u_0)$ ) in the ball; then, for  $t \geq T_*$  we have

$$\begin{aligned} \text{dist}_V(u(t), \widetilde{\mathcal{M}}) &= \inf_{v \in V} \|u(t) - v\| \leq \|u(t) - (P_m u(t) + \Phi_{\text{app}}(P_m u(t)))\| \\ &\leq \|Q_m u(t) - \Phi_{\text{app}}(P_m u(t))\| \leq \varepsilon. \end{aligned}$$

Consider the approximate inertial form based on  $\Phi_{\text{app}}$ ,

$$(3.4a) \quad \begin{aligned} \frac{dy_m}{dt} + \nu Ay_m + P_m C(y_m + \Phi_{\text{app}}(y_m)) \\ + P_m B(y_m + \Phi_{\text{app}}(y_m), y_m + \Phi_{\text{app}}(y_m)) = P_m f, \end{aligned}$$

or eventually the slightly different form

$$(3.4b) \quad \begin{aligned} \frac{dy_m}{dt} + \nu Ay_m + P_m C(y_m + \Phi_{\text{app}}(y_m)) \\ + P_m B(y_m + \Phi_{\text{app}}(y_m), y_m) + P_m B(y_m, \Phi_{\text{app}}(y_m)) = P_m f. \end{aligned}$$

The following theorem gives an error estimate in the norm of  $H$  for the nonlinear Galerkin method.

**Theorem 3.1.** *Under assumptions (3.1)–(3.3), let  $u(t)$  be a solution of (2.2), (2.13) such that*

$$(3.5) \quad \|u(t)\| \leq M_1 \quad \text{for all } t \geq 0.$$

*Suppose that  $y_m(t)$  is a solution of (3.4a) (respectively of (3.4b)) which satisfies  $y_m(0) = P_m u_0$  and*

$$(3.6) \quad \|y_m(t)\| \leq M_1 \quad \text{for all } t \geq 0.$$

*Then*

$$(3.7) \quad \begin{aligned} |u(t) - (y_m(t) + \Phi_{\text{app}}(y_m(t)))|^2 \\ \leq \left[ (1 + 2l^2) \int_0^t e^{\int_s^t A_m(\tau) d\tau} B_m(s) ds + \frac{2}{L_m} \right] \frac{\varepsilon^2 L_m}{\lambda_{m+1}}, \end{aligned}$$

*where  $L_m$ ,  $A_m$ , and  $B_m$  are given by (3.10), (3.12), and (3.13), respectively.*

**Remark 3.2.** The condition (3.5) means that we approximate solutions lying inside the absorbing ball  $B_V(0, M_1)$ . Since all solutions enter in finite time in this ball, and we are interested in long-time integration, this is not a serious restriction. Besides, the general case (any  $u_0$  sufficiently regular) can be handled in the same manner if one allows the constants  $l$ ,  $M_1$ , and  $\varepsilon$  in (3.1), (3.2), (3.3) to depend on  $u_0$ ; one can then show that the estimate (3.7) holds.

*Proof of Theorem 3.1.* We denote  $p = P_m u$ ,  $q = Q_m u$ ,  $y = y_m$ ,  $v = y + \Phi_{\text{app}}(y)$ ,  $\delta = p - y$ ,  $\Delta_1 = q - \Phi_{\text{app}}(p)$ ,  $\Delta_2 = \Phi_{\text{app}}(p) - \Phi_{\text{app}}(y)$ , and  $\Delta = \Delta_1 + \Delta_2$ .

We will only consider (3.4a); the same treatment works for (3.4b).

From (2.2) and (3.4a) we have

$$(3.8) \quad \frac{d\delta}{dt} + \nu A\delta + P_m C\delta + P_m (B(u, u) - B(v, v)) = 0.$$

By taking the inner product of (3.8) with  $\delta$  and using (2.4) and (2.6), we conclude that

$$\frac{1}{2} \frac{d|\delta|^2}{dt} + \alpha \|\delta\|^2 \leq |(B(\delta, u), \delta)| + |(B(\Delta, u), \delta)| + |(B(v, \Delta), \delta)|.$$

This inequality together with (2.7) and (2.6) yield

$$\begin{aligned} \frac{1}{2} \frac{d|\delta|^2}{dt} + \alpha \|\delta\|^2 \leq c_3 |\delta| \|\delta\| \|u\| + |(B(\Delta_1, u), \delta)| + |(B(\Delta_2, u), \delta)| \\ + |(B(y, \delta), \Delta_1)| + |(B(y, \delta), \Delta_2)| + |(B(\Phi_{\text{app}}(y), \delta), \Delta)|. \end{aligned}$$

Then, using (2.7), (2.9), and (2.10), we get

$$\begin{aligned}
 \frac{1}{2} \frac{d|\delta|^2}{dt} + \alpha \|\delta\|^2 &\leq c_3 |\delta| \|\delta\| \|u\| + c_5 |\Delta_1| \|u\| \|\delta\| L_m^{1/2} \\
 &\quad + c_3 |\Delta_2|^{1/2} \|\Delta_2\|^{1/2} \|u\| |\delta|^{1/2} \|\delta\|^{1/2} \\
 (3.9) \quad &\quad + c_5 \|y\| \|\delta\| |\Delta_1| L_m^{1/2} \\
 &\quad + c_3 |y|^{1/2} \|y\|^{1/2} \|\delta\| |\Delta_2|^{1/2} \|\Delta_2\|^{1/2} \\
 &\quad + c_3 |\Phi_{\text{app}}(y)|^{1/2} \|\Phi_{\text{app}}(y)\|^{1/2} \|\delta\| |\Delta|^{1/2} \|\Delta\|^{1/2},
 \end{aligned}$$

where

$$(3.10) \quad L_m = \left( 1 + \log \frac{\lambda m}{\lambda_1} \right).$$

Because of (3.1) and (3.3), inequality (3.9) gives

$$\begin{aligned}
 \frac{1}{2} \frac{d|\delta|^2}{dt} + \alpha \|\delta\|^2 &\leq c_3 |\delta| \|\delta\| \|u\| + \frac{c_5 \varepsilon}{\lambda_{m+1}^{1/2}} \|u\| \|\delta\| L_m^{1/2} + c_3 l \|\delta\| \|u\| |\delta| \\
 &\quad + \frac{c_5 \varepsilon}{\lambda_{m+1}^{1/2}} \|y\| \|\delta\| L_m^{1/2} + c_3 l |y|^{1/2} \|y\|^{1/2} \|\delta\|^{3/2} |\delta|^{1/2} \\
 &\quad + \frac{c_3 \varepsilon}{\lambda_{m+1}^{1/2}} \|\Phi_{\text{app}}(y)\| \|\delta\| + \frac{c_3 l}{\lambda_{m+1}^{1/2}} \|\Phi_{\text{app}}(y)\| \|\delta\|^2.
 \end{aligned}$$

By Young's inequality,

$$\begin{aligned}
 \frac{1}{2} \frac{d|\delta|^2}{dt} + \alpha \|\delta\|^2 &\leq \left( \frac{3c_3^2}{\alpha} \|u\|^2 + \frac{3}{\alpha} c_3^2 l^2 \|u\|^2 + \frac{27c_4^4 l^4}{4\alpha^3} |y|^2 \|y\|^2 \right) |\delta|^2 \\
 (3.11) \quad &\quad + \frac{3}{4} \alpha \|\delta\|^2 + \frac{3c_5^2 \varepsilon^2}{\alpha \lambda_{m+1}} \|u\|^2 L_m + \frac{3c_5^2 \varepsilon^2}{\alpha \lambda_{m+1}} \|y\|^2 L_m \\
 &\quad + \frac{3c_3^2 \varepsilon^2}{\alpha \lambda_{m+1}} \|\Phi_{\text{app}}(y)\|^2 + \frac{3c_3^2 l^2}{\alpha} \|\Phi_{\text{app}}(y)\|^2 |\delta|^2.
 \end{aligned}$$

Denote

$$(3.12) \quad A_m(t) = \frac{6c_3^2}{\alpha} \|u\|^2 + \frac{6c_3^2 l^2}{\alpha} \|u\|^2 + \frac{27c_4^4 l^4}{2\alpha^3} |y|^2 \|y\|^2 + \frac{6c_3^2 l^2}{\alpha} \|\Phi_{\text{app}}(y)\|^2$$

and

$$(3.13) \quad B_m(t) = \frac{6c_5^2}{\alpha} \|u\|^2 + \frac{6c_5^2}{\alpha} \|y\|^2 + \frac{6c_3^2}{\alpha L_m^2} \|\Phi_{\text{app}}(y)\|^2.$$

Then, by Gronwall's Lemma, and since  $\delta(0) = 0$ , (3.11) gives

$$(3.14) \quad |\delta(t)|^2 \leq \frac{\varepsilon^2 L_m}{\lambda_{m+1}} \int_0^t e^{\int_s^t A_m(\tau) d\tau} B_m(s) ds.$$

This inequality together with (3.14), (3.1), and (3.3) yield (3.7).  $\square$

*Remark 3.3.* The time-dependent term in the right-hand side of (3.7) can be evaluated easily from the a priori estimates for (2.2) and (3.4). Moreover, notice that these estimates will not involve the term  $\sup_{t \geq 0} \|u(t)\|^2$  but its time average, which is much smaller. Also, notice that this estimate (3.7) does not involve at all the term  $|Au(t)|$ . We will refer to this observation later in Remark 4.1, when we will compare our estimates with those of Rautmann [44, 45].

In the next theorem we give an upper bound for the Dirichlet norm of the error.

**Theorem 3.3.** *Assume that  $u(t)$  and  $y(t)$  are as in Theorem 3.1. Then*

$$(3.15) \quad \|u(t) - (y(t) + \Phi_{\text{app}}(y(t)))\|^2 \leq \left( (1 + 2l) \int_0^t e^{\int_s^t \tilde{A}_m(\tau) d\tau} \tilde{B}_m(s) ds + 2 \right) \varepsilon^2,$$

where  $\tilde{A}_m$  and  $\tilde{B}_m$  are given in (3.17) and (3.18), respectively.

*Proof.* We will use the same notation as in Theorem 3.1. We take the inner product of (3.5) with  $A\delta$  and obtain

$$\begin{aligned} \frac{1}{2} \frac{d\|\delta\|^2}{dt} + \nu |A\delta|^2 &\leq |(C\delta, A\delta)| + |(B(\delta, u), A\delta)| + |(B(\Delta, u), A\delta)| \\ &\quad + |(B(v, \Delta), A\delta)| + |(B(v, \delta), A\delta)|. \end{aligned}$$

From (2.5), (2.12), and (2.7) we have

$$\begin{aligned} \frac{1}{2} \frac{d\|\delta\|^2}{dt} + \nu |A\delta|^2 &\leq K\|\delta\| |A\delta| + c_6\|\delta\| \|u\| \left( 1 + \log \frac{|Au|^2}{\|u\|^2 \lambda_1} \right)^{1/2} |A\delta| \\ &\quad + c_3|\Delta|^{1/2} \|\Delta\|^{1/2} \|u\| |A\delta|^{1/2} |A^{3/2}\delta|^{1/2} \\ &\quad + |(B(y, \Delta), A\delta)| + |(B(\Phi_{\text{app}}(y), \Delta), A\delta)| \\ &\quad + |(B(y, \delta), A\delta)| + |(B(\Phi_{\text{app}}(y), \delta), A\delta)|. \end{aligned}$$

Applying (2.10) and (2.7), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d\|\delta\|^2}{dt} + \nu |A\delta|^2 &\leq K\|\delta\| |A\delta| + c_6\|\delta\| \|u\| \left( 1 + \log \frac{|Au|^2}{\|u\|^2 \lambda_1} \right)^{1/2} |A\delta| \\ &\quad + c_3\|\Delta\| \|u\| |A\delta| + c_5\|y\| \left( 1 + \log \frac{|Ay|^2}{\|y\|^2 \lambda_1} \right)^{1/2} \\ &\quad \times (\|\delta\| + \|\Delta\|) |A\delta| \\ &\quad + c_3 |\Phi_{\text{app}}(y)|^{1/2} \|\Phi_{\text{app}}(y)\|^{1/2} (\|\delta\| + \|\Delta\|) |A\delta|^{1/2} |A^{3/2}\delta|^{1/2}. \end{aligned}$$

From (3.1) and (3.3) we have

$$\begin{aligned} \frac{1}{2} \frac{d\|\delta\|^2}{dt} + \nu |A\delta|^2 &\leq K\|\delta\| |A\delta| + c_6\|\delta\| \|u\| \left( 1 + \log \frac{|Au|^2}{\|u\|^2 \lambda_1} \right)^{1/2} |A\delta| \\ &\quad + c_5\|y\| \left( 1 + \log \frac{|Ay|^2}{\|y\|^2 \lambda_1} \right)^{1/2} [(1+l)\|\delta\| + \varepsilon] |A\delta| \\ &\quad + c_3(\varepsilon + l\|\delta\|) \|u\| |A\delta| + c_3[(1+l)\|\delta\| + \varepsilon] \|\Phi_{\text{app}}(y)\| |A\delta|. \end{aligned}$$

By Young's inequality we get

$$(3.16) \quad \frac{d\|\delta\|^2}{dt} + \frac{2}{3}\nu |A\delta|^2 \leq \tilde{A}_m(t) \|\delta\|^2 + \tilde{B}_m(t) \varepsilon^2,$$

where

$$(3.17) \quad \begin{aligned} \tilde{A}_m(t) &= \frac{6K^2}{\nu} + \frac{6c_6^2}{\nu} \|u\|^2 \left( 1 + \log \frac{|Au|^2}{\|u\|^2 \lambda_1} \right) + \frac{6c_3^2 l^2}{\nu} \|u\|^2 \\ &\quad + \frac{6c_5^2 (1+l)^2}{\nu} \|y\|^2 \left( 1 + \log \frac{|Ay|^2}{\|y\|^2 \lambda_1} \right) + \frac{6c_3^2 (1+l)^2}{\nu} \|\Phi_{\text{app}}(y)\|^2 \end{aligned}$$



and

$$(3.18) \quad \tilde{B}_m(t) = \frac{6c_3^2}{\nu} \|u\|^2 + \frac{6c_5^2}{\nu} \|y\|^2 \left( 1 + \log \frac{|Ay|^2}{\|y\|^2 \lambda_1} \right) + \frac{6c_3^2}{\nu} \|\Phi_{\text{app}}(y)\|^2.$$

By Gronwall's Lemma, (3.16) implies

$$(3.19) \quad \|\delta(t)\|^2 \leq \left( \int_0^t e^{\int_s^t \tilde{A}_m(\tau) d\tau} \tilde{B}_m(s) ds \right) \varepsilon^2.$$

Therefore, (3.15) follows from (3.19), (3.1), and (3.3).  $\square$

*Remark 3.4.* Here again, we remark that the error estimate in (3.15) involves only the logarithm of the  $|Au|$  norm of the solution, not  $|Au|$ .

**3.1. Stability and uniform error bounds in time.** It is observed that the error bounds in (3.7) and (3.15) are growing exponentially in time. Since certain solutions of turbulent flows for high Reynolds numbers are unstable, one should expect such growth in the bounds. Indeed, if, for instance,  $u(t) \equiv u_s$  is an unstable hyperbolic steady state, then it is natural to observe, at least for a short time, exponential growth of certain small perturbations about  $u_s$ . If, on the other hand, one assumes that the exact solution  $u(t)$  is exponentially stable, as in Heywood [28] and Heywood and Rannacher [29], then one can get a time-independent upper bound for the error in (3.7) and (3.15), without changing the accuracy in the space discretization. We will report the details of this result elsewhere.

#### 4. APPLICATIONS

**4.1. The Galerkin method.** In the classical Galerkin method, the equations are projected on the linear space  $\tilde{\mathcal{M}}_0 = P_m H$ . This linear manifold can be viewed as the simplest AIM.

More precisely, let us consider equation (2.2) with  $C \equiv 0$ , and let  $u_0$  be an initial value such that  $|u_0| \leq R_0$ ,  $\|u_0\| \leq R_1$ . There exists a time  $T_0$ , which depends on  $R_0$ ,  $R_1$ , and the data  $(\nu, |f|, \lambda_1)$ , such that

$$(4.1) \quad |u(t)| \leq M_0, \quad \|u(t)\| \leq M_1 \quad \text{for all } t \geq T_0,$$

where  $M_0, M_1$  are independent of  $u_0$  but depend on  $(\nu, |f|, \lambda_1)$  (see for instance Constantin and Foias [5] and Temam [47]). Alternatively, (4.1) means that the ball  $B_H(0, M_0)$  (respectively  $B_V(0, M_1)$ ) is an absorbing set in  $H$  (respectively in  $V$ ).

According to Foias et al. [13, 14], the time  $T_0$  can be chosen so that  $q_m = Q_m u(t)$  satisfies

$$(4.2) \quad \|q_m(t)\| \leq K L_m^{1/2} \lambda_{m+1}^{-1/2} \quad \text{for all } t \geq T_0.$$

Here,  $K$  depends only on the data  $(\nu, |f|, \lambda_1)$ , and  $L_m$  is given by (3.10). Since  $\|q_m(t)\| = \text{dist}_V(u(t), P_m H)$ , (4.2) yields that  $\tilde{\mathcal{M}}_0 = P_m H$  is an AIM of order  $\varepsilon = K L_m^{1/2} \lambda_{m+1}^{-1/2}$ . Obviously, for  $\tilde{\mathcal{M}}_0$ , one has  $\Phi_{\text{app}} \equiv 0$ .

The approximate inertial form associated with  $\tilde{\mathcal{M}}_0$  is the standard Galerkin procedure for (2.1):

$$(4.3) \quad \frac{dy_m}{dt} + \nu Ay_m + P_m R(y_m) = 0.$$

The a priori estimates for (4.3) are well known. In particular, the following uniform-in-time bound can be derived:

$$\|y_m(t)\| \leq M_1 \quad \text{for all } t \geq T_0,$$

where  $T_0$  is as above.

This shows (3.3). Theorems 3.1 and 3.3 apply and provide the estimates

$$\begin{aligned} |u(t) - y_m(t)|^2 &\leq \left( \int_0^t e^{\int_s^t A_m(\tau) d\tau} B_m(s) ds + \frac{2}{L_m} \right) \frac{L_m^2}{\lambda_{m+1}^2}, \\ \|u(t) - y_m(t)\|^2 &\leq \left( 2 + \int_0^t e^{\int_s^t \tilde{A}_m(\tau) d\tau} \tilde{B}_m(s) ds \right) \frac{L_m}{\lambda_{m+1}}, \end{aligned}$$

where  $L_m$ ,  $A_m$ ,  $B_m$ ,  $\tilde{A}_m$ , and  $\tilde{B}_m$  are given by (3.10), (3.12), (3.13), (3.17), and (3.18), respectively.

*Remark 4.1.* The above estimates are consistent with the convergence result of Foias [11]. In comparison with the error bounds of Rautmann [44, 45], we improve in the time-dependent term. This is achieved, as we observed in Remarks 3.3 and 3.4, by not involving  $|Au(t)|$  or its time average, unlike Rautmann [44, 45]. Because, even in the case of homogeneous boundary conditions, the bounds available for  $|Au(t)|$  depend exponentially on a power of the Reynolds number (see, for instance, Constantin and Foias [5] and Foias and Temam [20]). On the other hand, in establishing this improvement we pick up the extra  $L_m$ -term in the spatial error, which is only logarithmic in  $\lambda_m$ . Moreover, in view of the example of Titi [55, 56], and provided  $f \in L^2$ , the above estimates are sharp, asymptotically in  $m$ , as  $m \rightarrow \infty$ , up to a logarithmic term. This observation makes the nonlinear Galerkin methods, below, more accurate than the usual Galerkin method. It is worth mentioning, however, that if the forcing term in (2.1) with homogeneous or periodic boundary conditions, is more regular, then one can improve the estimate (4.2) and consequently the above estimates (see Jones and Titi [34]). A similar observation is valid for nonlinear Galerkin methods.

**4.2. Nonlinear Galerkin methods.** The above AIM,  $\tilde{\mathcal{M}}_0$ , is a linear space. The simplest nonlinear AIM,  $\tilde{\mathcal{M}}_1 = \text{graph } \Phi_1$ , is given by the resolution of the Stokes problem (with homogeneous or periodic boundary conditions)

$$\nu Aq + Q_m R(p) = 0, \quad \text{i.e., } \Phi_1(p) = -(\nu A)^{-1} Q_m R(p).$$

According to Foias et al. [13, 14], the following estimate holds:

$$\|Q_m u(t) - \Phi_1(P_m u(t))\| \leq KL_m \lambda_{m+1}^{-1} \quad \text{for all } t \geq T_0,$$

where  $T_0$  is as above. This shows that  $\tilde{\mathcal{M}}_1$  is an AIM of order  $\varepsilon = KL_m \lambda_{m+1}^{-1}$ . In particular,  $\tilde{\mathcal{M}}_1$  is of better order than  $\tilde{\mathcal{M}}_0$ .

The nonlinear Galerkin method (3.4b) associated with  $\tilde{\mathcal{M}}_1$  is introduced in Marion and Temam [41], while (3.4a) is studied in Devulder and Marion [8]. In both cases, the existence of a constant  $M$  depending only on  $(\nu, |f|, \lambda_1)$  such that

$$\|y_m(t)\| \leq M \quad \text{for all } t \geq T_0$$

is derived (in the case of (3.4b) the above holds provided  $m$  is large enough, depending on  $u_0$ ). This guarantees (3.3) and, together with other a priori estimates, yield the following convergence results:

$$(4.4) \quad y_m + \Phi_1(y_m) \rightarrow u \text{ as } m \rightarrow \infty \text{ in } L^2(0, T; D(A)) \text{ and } L^p(0, T; V) \text{ strongly for all } T > 0 \text{ and all } 1 \leq p < +\infty,$$

and

$$(4.5) \quad y_m + \Phi_1(y_m) \rightarrow u \text{ as } m \rightarrow \infty \text{ in } L^\infty(\mathbf{R}^+; V) \text{ weak-star.}$$

Here, the initial data  $u_0$  is assumed to be given in  $V$  (see Marion and Temam [41] and Devulder and Marion [8]).

The Lipschitz property (3.1) for  $\Phi_1$  with  $l = K\lambda_{m+1}^{-1/4}$  is proved in the Appendix.

Theorems 3.1 and 3.3 apply here and give the following error bounds for the nonlinear Galerkin approximation induced by  $\Phi_1$ :

$$(4.6) \quad \begin{aligned} & |u(t) - (y_m(t) + \Phi_1(y_m(t)))|^2 \\ & \leq \left[ \left( 1 + \frac{K}{\lambda_{m+1}^{1/2}} \right) \int_0^t e^{\int_s^t A_m(\tau) d\tau} B_m(s) ds + \frac{2}{L_m} \right] \frac{KL_m^3}{\lambda_{m+1}^3}, \end{aligned}$$

$$(4.7) \quad \begin{aligned} & \|u(t) - (y_m(t) + \Phi_1(y_m(t)))\|^2 \\ & \leq \left[ \left( 1 + \frac{K}{\lambda_{m+1}^{1/4}} \right) \int_0^t e^{\int_s^t \tilde{A}_m(\tau) d\tau} \tilde{B}_m(s) ds + 2 \right] \frac{KL_m^2}{\lambda_{m+1}^2} \end{aligned}$$

with  $L_m, A_m, B_m, \tilde{A}_m$ , and  $\tilde{B}_m$  respectively given by (3.10), (3.12), (3.13), (3.17), and (3.18). It follows from (4.6), (4.7) that the order of the spatial discretization is improved by comparison with the Galerkin method (see §4.1 above).

Another simple AIM was introduced by Foias et al. [18], known as the Euler-Galerkin AIM, and which was used in real computations by Foias et al. [12] and Jolly et al. [32, 33]. Nonlinear Galerkin methods of the type (3.4), induced by the Euler-Galerkin AIM, were applied to the Kuramoto-Sivashinsky equation and studied analytically as well as computationally in Jolly et al. [32]. More involved AIM have been introduced in Foias and Temam [21], Temam [50]. In particular, in Temam [50] (see also Debussche and Marion [7]), a method for constructing a sequence of manifolds  $\tilde{\mathcal{M}}_j = \text{graph } \Phi_j$ , which is providing better and better orders of approximation as  $j$  increases, is presented. At step  $j$ ,  $\tilde{\mathcal{M}}_j$  is such that

$$(4.8) \quad \|Q_m u(t) - \Phi_j(P_m u(t))\| \leq K_j L_m^{j/2+1/2} \lambda_{m+1}^{-j/2-1/2} \quad \text{for all } t \geq T_0,$$

where  $K_j$  depends on  $j$  and on the data  $(\nu, |f|, \lambda_1)$ . The precise definition of the  $\Phi_j$ 's is recalled in the Appendix, where the property (3.1) is proved with  $l = l_j = K'_j \lambda_{m+1}^{-1/4}$ , where  $K'_j$  depends on  $j$  and the data  $(\nu, |f|, \lambda_1)$ . It is clear from (4.8) that as  $j$  increases,  $\tilde{\mathcal{M}}_j$  provides a better-order approximation to the solution: for some fixed  $m$ , the solutions may not be closer to  $\tilde{\mathcal{M}}_{j'}$ , for  $j' > j$ , than to  $\tilde{\mathcal{M}}_j$ , but this holds for  $m$  large enough.

The associated nonlinear Galerkin method (3.4a), is considered in Devulder and Marion [8], where the estimate (3.3) is proved and convergence results similar to (4.4), (4.5) are derived.

Therefore, one can apply Theorems 3.1 and 3.3 and obtain the following error bounds:

$$\begin{aligned}
 & |u(t) - (y_m(t) + \Phi_j(y_m(t)))|^2 \\
 & \leq \left[ \left( 1 + \frac{K_j'^2}{\lambda_{m+1}^{1/2}} \right) \int_0^t e^{\int_s^t A_m(\tau) d\tau} B_m(s) ds + \frac{2}{L_m} \right] \frac{K_j^2 L_m^{j+2}}{\lambda_{m+1}^{j+1}}, \\
 & \|u(t) - (y_m(t) + \Phi_j(y_m(t)))\|^2 \\
 & \leq \left[ \left( 1 + \frac{K_j'}{\lambda_{m+1}^{1/4}} \right) \int_0^t e^{\int_s^t \tilde{A}_m(\tau) d\tau} \tilde{B}_m(s) ds + 2 \right] \frac{K_j^2 L_m^{j+1}}{\lambda_{m+1}^{j+1}},
 \end{aligned}$$

where  $L_m, A_m, B_m, \tilde{A}_m,$  and  $\tilde{B}_m$  are given by (3.10), (3.12), (3.13), (3.17), and (3.18), respectively. Clearly, for increasing  $j$ , the order with respect to  $m$  of these estimates improves.

**4.3. Other nonlinear Galerkin methods.** The above methods correspond to manifolds whose equations are explicit with respect to the unknowns. Here we present numerical schemes associated with implicit AIMs.

The first manifold that we consider is the analytic manifold  $\mathcal{M}^s$ , which contains all the stationary solutions of (2.2). Denote  $B_m = P_m B_V(0, M_1)$ , where  $M_1$  satisfies (4.1). According to Foias and Temam [19], Foias and Saut [16], and Titi [55, 56], for  $m$  large enough, there exists a mapping  $\Phi^s: \mathcal{B}_m \rightarrow Q_m V$  which satisfies

$$\begin{aligned}
 (4.9) \quad \Phi^s(p) = (\nu A)^{-1} (Q_m f - Q_m C(p + \Phi^s(p)) \\
 - Q_m B(p + \Phi^s(p), p + \Phi^s(p))),
 \end{aligned}$$

for all  $p \in \mathcal{B}_m$ .

The manifold  $\mathcal{M}^s = \text{graph } \Phi^s$  has been studied from the point of view of AIMs in Titi [55, 56], where it is shown to be of the order of  $\varepsilon = KL_m^{1/2} \lambda_{m+1}^{-3/2}$ . Moreover, Jolly et al. [32, 33] used this manifold to demonstrate the computational efficiency of the nonlinear Galerkin method (see also Graham et al. [25]).

In Titi [55, 56] it is shown that  $\Phi^s$  is a Lipschitz manifold. Using the usual energy estimates, one can easily show that (3.4a) has a global solution for initial data in  $\mathcal{B}_m$ . Such analysis has been carried out in Jolly et al. [33] for the Kuramoto-Sivashinsky equation. In the case of (3.4b), using techniques similar to the ones in Devulder and Marion [8], one can show the existence of a constant  $M_0$  depending only on  $(\nu, |f|, \lambda_1)$  and an integer  $\tilde{m}$  depending on  $(\nu, |f|, \lambda_1)$  and  $u_0$  (through  $\|u_0\|$ ) such that, if  $m \geq \tilde{m}$ , problem (3.4) together with  $y_m(0) = P_m u_0$  possesses a unique solution  $y_m(t)$  defined for all  $t > 0$  with

$$(4.10) \quad \|y_m(t)\| \leq M_0 \quad \text{for all } t \geq 0.$$

This guarantees (3.2) (with  $M_1$  replaced by  $M_0$ ). The estimate (4.10) enables us to derive convergence results analogous to (4.4) and (4.5).

Finally, it is easy to derive the Lipschitz property (3.1) for  $\Phi^s$  with  $l = KL_m^{1/2}\lambda_{m+1}^{-1/2}$ .

Theorems 3.1 and 3.3 apply and yield the following error bounds for the nonlinear Galerkin scheme:

$$\begin{aligned}
 & |u(t) - (y_m(t) + \Phi^s(y_m(t)))|^2 \\
 & \leq \left[ \left( 1 + \frac{KL_m}{\lambda_{m+1}} \right) \int_0^t e^{\int_s^t A_m(\tau) d\tau} B_m(s) ds + \frac{2}{L_m} \right] \frac{KL_m^2}{\lambda_{m+1}^4}, \\
 & \|u(t) - (y_m(t) + \Phi^s(y_m(t)))\|^2 \\
 & \leq \left[ \left( 1 + \frac{KL_m^{1/2}}{\lambda_{m+1}^{1/2}} \right) \int_0^t e^{\int_s^t \tilde{A}_m(\tau) d\tau} \tilde{B}_m(s) ds + 2 \right] \frac{KL_m}{\lambda_{m+1}^3}.
 \end{aligned}$$

A more involved implicit AIM is given by the resolution of the system

$$\begin{aligned}
 & p_0^1 + \nu Ap + P_m C(p + q) + P_m B(p + q, p + q) = P_m f, \\
 (4.11) \quad & \nu Aq_1^1 + Q_m Cp_0^1 + Q_m (B(p_0^1, p + q) + B(p + q, p_0^1)) = 0, \\
 & q_1^1 + \nu Aq + Q_m C(p + q) + Q_m B(p + q, p + q) = Q_m f.
 \end{aligned}$$

According to Devulder and Marion [8], for every  $p \in \mathcal{B}_m$  and  $m$  large enough, (4.11) possesses a unique solution  $(p_0^1, q_1^1, q)$  with  $p_0^1 \in P_m V$ ,  $q_1^1 \in Q_m V$ , and  $q \in Q_m B_V(0, M_1)$ . We set  $q = \tilde{\Phi}(p)$ . Then,  $\tilde{\mathcal{M}} = \text{graph } \tilde{\Phi}$  is an AIM of order  $\varepsilon = KL_m \lambda_{m+1}^{-2}$ . The corresponding nonlinear Galerkin method (3.4) is studied in Devulder and Marion [8], where property (3.6) is checked as well as convergence results analogous to (4.4), (4.5).

Lastly, property (3.1) holds with  $l = K\lambda_{m+1}^{-1/4}$ . Therefore, by applying Theorems 3.1 and 3.3, we obtain the following error bounds for the nonlinear Galerkin methods:

$$\begin{aligned}
 & |u(t) - (y_m(t) + \tilde{\Phi}(y_m(t)))|^2 \\
 & \leq \left[ \left( 1 + \frac{K}{\lambda_{m+1}^{1/2}} \right) \int_0^t e^{\int_s^t A_m(\tau) d\tau} B_m(s) ds + \frac{2}{L_m} \right] \frac{KL_m^3}{\lambda_m^5}, \\
 & \|u(t) - (y_m(t) + \tilde{\Phi}(y_m(t)))\|^2 \\
 & \leq \left[ \left( 1 + \frac{K}{\lambda_{m+1}^{1/4}} \right) \int_0^t e^{\int_s^t \tilde{A}_m(\tau) d\tau} \tilde{B}_m(s) ds + 2 \right] \frac{KL_m^2}{\lambda_{m+1}^5}.
 \end{aligned}$$

**4.4. Remarks on the discretization of the manifolds.** The methods (3.4a) or (3.4b) involve the equation  $z = \Phi_{\text{app}}(y)$  of the manifold. Generally speaking, in order to compute  $\Phi_{\text{app}}$ , we need to compute infinitely many Fourier coefficients. This prevents us from directly using the schemes in real computations. Here we make some remarks on the discretization of the equation of the manifold.

First we note that from a practical point of view one can mainly use the schemes in the case of periodic boundary conditions where the eigenfunctions of the Stokes operator are known. They correspond to the usual Fourier modes. Then it is easy to check that if  $p \in P_m V$ , one has  $B(p, p) \in P_{\alpha(m)} V$ , for some  $\alpha(m) < \infty$ . Consequently, all equations for implicit AIMs in §4.2 turn out

to be discrete equations. For example, the equation for  $\mathcal{M}_j$  is an equation in  $P_{\alpha(m)}V$ . Therefore, one can replace without any change in (3.4a), (3.4b),  $\Phi_{\text{app}}$  by  $(P_{\alpha(m)} - P_m)\Phi_{\text{app}}$ , where the function  $\alpha(m)$  depends on the AIM under consideration. In particular, the error estimates are the same as above.

For implicit manifolds the situation is less straightforward. However, these manifolds are constructed using the contraction principle. Therefore, they can be approximated by simple explicit functions, thanks to the successive approximations procedure and, as above, the approximating equations turn out to be discrete equations. Estimates of the error in such approximations can be found in Titi [55, 56].

In the case of general boundary conditions (which is not realistic from a practical computational point of view, as already noted), the equations of the manifolds can still be discretized by using  $N$  modes, but  $N$  must be large enough so that the error due to this discretization is of the same order as the error of the nonlinear Galerkin method. For example, for the manifold  $\Phi^s$  in (4.9), it is shown in Titi [56] that  $N$  must be of the order  $m^3$ .

APPENDIX. PROOF OF THE LIPSCHITZ PROPERTY FOR THE MANIFOLD  $\mathcal{M}_j$

Our aim in this appendix is to prove the Lipschitz condition (3.1) for the nonlinear Galerkin methods presented in Devulder and Marion [8]. For the sake of simplicity we consider the case  $C \equiv 0$ .

These methods are associated with the approximate inertial manifolds  $\mathcal{M}_j$  constructed in Temam [50], a construction which we will now recall. The manifolds are defined recursively; they are obtained as graphs of functions  $\Phi_j$  mapping  $P_mV$  into  $Q_mV$ .

The first manifold,  $q_1 = \Phi_1(p)$ , satisfies (see §4.2)

$$(A.1) \quad \nu Aq_1 + Q_mB(p) = Q_mf, \quad \text{where } B(p) = B(p, p).$$

Then,  $q_2 = \Phi_2(p)$  is the solution of

$$(A.2) \quad \nu Aq_2 + Q_mB(p + q_1) = Q_mf,$$

where  $q_1$  is given by (A.1). Generally, at step  $j$ , for  $j \geq 3$ ,  $q_j = \Phi_j(p)$  is given by

$$(A.3) \quad q_{j-2}^1 + \nu Aq_j + Q_mB(p + q_{j-1}) = Q_mf.$$

Here, the  $q_j^1$  are approximations of  $\frac{dq}{dt}$ , where  $q = Q_mu$ . To define them, we must introduce two families  $q_j^i, p_j^i$ , where the  $q_j^i$  approximate in some sense  $\frac{dq}{dt}$ , while the  $p_j^i$  approximate  $\frac{dp}{dt}$ , with  $p = P_mu$ . At the step  $j$ , i.e., when we want to determine  $q_j$  from (A.3) ( $j \geq 3$ ), we construct the sequence

$$(A.4) \quad p_{j-1-2i}^i, \quad i = 0, \dots, [(j-1)/2],$$

for increasing values of  $i$ , and then the sequence

$$(A.5) \quad q_{j-2i}^i, \quad i = [(j+1)/2], \dots, 1, 0,$$

in decreasing order of  $i$ . Of course, at that stage, the similar sequences with  $j$  replaced by  $k \leq j-1$  are already defined. For  $i = 0$ , we have  $p_{j-1}^0 = p$ ,

$q_j^0 = q_j$  for all  $j$ . And we also set for convenience  $q_{(-1)}^k = q_0^k = 0$  for all  $k$ . The sequence (A.4) is determined by the following recursive formula:

$$(A.6)_{j,i} \quad \sum_{k=0}^i \binom{i}{k} P_m B(p_{j-1-2k}^k + q_{j-1-2k}^k, p_{j-1+2i-2k}^{i-k} + q_{j-1+2i-2k}^{i-k}) + p_{j-3-2i}^{i+1} + \nu A p_{j-1-2i}^i = \begin{cases} P_m f & \text{for } i = 0, \\ 0 & \text{for } i \geq 1, \end{cases}$$

which gives explicitly  $p_{j-3-2i}^{i+1}$  for  $i = 0, \dots, [(j-1)/2]-1$ . Then the sequence (A.5) is defined by

$$q_{j-2i} = 0 \quad \text{for } i = [(j+1)/2],$$

$$(A.7)_{j,i} \quad \sum_{k=0}^i \binom{i}{k} Q_m B(p_{j-1-2k}^k + q_{j-1-2k}^k, p_{j-1+2i-2k}^{i-k} + q_{j-1+2i-2k}^{i-k}) + q_{j-2-2i}^{i+1} + \nu A q_{j-2i}^i = \begin{cases} P_m f & \text{for } i = 0, \\ 0 & \text{for } i \geq 1. \end{cases}$$

The order of the manifold  $q_j = \Phi_j(p)$  is given by the estimate (4.8) proved in Devulder and Marion [8].

Here, we seek to prove the Lipschitz property (3.1) for the  $\mathcal{M}_j$ 's. We first need the following lemma, which gives in particular estimates on the nonlinear operator  $\Phi_j$  valid in the absorbing ball  $B_V(0, M_1)$  for (2.2).

**Lemma A.1.** *Let  $j \geq 1$  and let  $p \in P_m B_V(0, M_1)$ . Then, for  $1 \leq k \leq j$ , the families (A.4) and (A.5) satisfy the following estimates:*

(i) *For  $i = 1, \dots, [(k-1)/2]$ , the term  $p_{k-1-2i}^i$  given by (A.6) $_{k,i}$  satisfies*

$$(A.8)_{k,i} \quad |p_{k-1-2i}^i| \leq K_{k,i} M_1^{\alpha_{k,i}} \lambda_{m+1}^{i-1/2},$$

where  $K_{k,i} \geq 0$  denotes a constant depending on the data  $(\nu, |f|, \lambda_1)$  and on  $k, i$ , while  $\alpha_{k,i} \geq 0$  depends only on  $k, i$ .

(ii) *For  $i = [(k+1)/2], \dots, 0$ , the function  $q_{k-2i}^i$  given by (A.7) $_{k,i}$  satisfies*

$$(A.9)_{k,i} \quad |A q_{k-2i}^i| \leq K'_{k,i} M_1^{\beta_{k,i}} \lambda_{m+1}^{i+1/4},$$

where  $K'_{k,i} \geq 0$  denotes a constant depending on the data,  $k$  and  $i$ , while  $\beta_{k,i} \geq 0$  depends only on  $k, i$ .

*Proof.* This lemma is a variant of Lemma 3.1 in Devulder and Marion [8], and its proof will be omitted.  $\square$

We will now show that  $\Phi_j$  is Lipschitz continuous. More precisely, we seek to show that the following estimates hold.

**Lemma A.2.** *Let  $j \geq 1$  and let  $p, \hat{p} \in P_m B_V(0, M_1)$ . Then, for  $1 \leq k \leq j$ , the families (A.4) and (A.5), defined for  $p$  and  $\hat{p}$ , are such that*

(i) *for  $i = 1, \dots, [(k-1)/2]$ ,*

$$(A.10)_{k,i} \quad |p_{k-1-2i}^i - \hat{p}_{k-1-2i}^i| \leq K_{k,i} M_1^{\gamma_{k,i}} \lambda_{m+1}^{i-1/2} \|p - \hat{p}\|;$$

(ii) *for  $i = [(k+1)/2], \dots, 0$ ,*

$$(A.11)_{k,i} \quad |A(q_{k-2i}^i - \hat{q}_{k-2i}^i)| \leq K'_{k,i} M_1^{\delta_{k,i}} \lambda_{m+1}^{i+1/4} \|p - \hat{p}\|,$$

where  $K_{k,i}$ ,  $K'_{k,i}$ , denote constants depending on the data,  $k$  and  $i$ , while  $\gamma_{k,i}$ ,  $\delta_{k,i}$  depend only on  $k$  and  $i$ .

*Remark A.3.* The inequality (A.11) $_{j,0}$  together with the spectral inequality (A.16) give

$$(A.12) \quad \|\Phi_j(p) - \Phi_j(\hat{p})\| \leq l_j \|p - \hat{p}\| \quad \text{for all } p, \hat{p} \in P_m B_V(0, M_1),$$

where

$$(A.13) \quad l_j = K'_{j,0} M_1^{\delta_{j,0}} \lambda_{m+1}^{-1/4},$$

which is the desired Lipschitz property.

*Proof of Lemma A.2.* In the sequel, we will need the following properties of the nonlinear operator  $B$  (see Constantin and Foias [5], Temam [47]):

$$(A.14) \quad |B(u, v)| \leq c_7 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} \quad \text{for all } u \in V \text{ and } v \in D(A),$$

$$(A.15) \quad |B(u, v)| \leq c_8 |u|^{1/2} |Au|^{1/2} \|v\| \quad \text{for all } u \in D(A) \text{ and } v \in V,$$

and the spectral estimates

$$(A.16) \quad \begin{aligned} \lambda_m^{-1/2} |Ap| &\leq \|p\| \leq \lambda_m^{1/2} |p| && \text{for all } p \in P_m H, \\ \lambda_1^{-1/2} |Au| &\geq \|u\| \geq \lambda_m^{1/2} |u| && \text{for all } u \in D(A), \\ \lambda_{m+1}^{-1/2} |Aq| &\geq \|q\| \geq \lambda_{m+1}^{1/2} |q| && \text{for all } q \in Q_m H. \end{aligned}$$

Let  $j \geq 1$  be fixed and let  $p, \hat{p} \in P_m B_V(0, M_1)$ . The proof of estimates (A.10) $_{k,i}$ , (A.11) $_{k,i}$  will rely on an induction argument on  $k$ ,  $1 \leq k \leq j$ .

(a) *Initialization of the induction* ( $k = 1$ ). For  $k = 1$ , there is nothing to prove as far as (A.10) is concerned. Next, (A.11) reduces to an estimate for  $(q_1^0 - \hat{q}_1^0)$ . But we have

$$\nu A(q_1^0 - \hat{q}_1^0) = Q_m(B(\hat{p}, \hat{p}) - B(p, p)),$$

and therefore

$$\nu |A(q_1^0 - \hat{q}_1^0)| \leq |B(p - \hat{p}, p)| + |B(\hat{p}, p - \hat{p})|.$$

Hence, using (A.15) and (A.16), we get

$$|A(q_1^0 - \hat{q}_1^0)| \leq \frac{2c_8 \lambda_m^{1/4}}{\nu \lambda_1^{1/4}} M \|p - \hat{p}\|,$$

so that (A.11) $_{1,0}$  holds.

(b) *The induction argument.* We now assume that estimates (A.10) $_{k,i}$  and (A.11) $_{k,i}$  have been proved up to level  $k - 1$ , and we seek to prove them at level  $k$ .

Following the determination of the families (A.4) and (A.5), we will first prove (A.10) $_{k,i}$  for increasing values of  $i$ , and then for (A.11) $_{k,i}$  by induction on  $i$  for decreasing values of  $i$ .

(b.1) *Proof of (A.10) $_{k,i}$ .* For  $i = 1$ , (A.10) $_{k,i}$  is an estimate for  $(p_{k-3}^1 - \hat{p}_{k-3}^1)$ . But we have

$$(p_{k-3}^1 - \hat{p}_{k-3}^1) = \nu A(\hat{p} - p) + P_m(B(\hat{p} + \hat{q}_{k-1}^0) - B(p + q_{k-1}^0)).$$



Hence,

$$\begin{aligned} |p_{k-3}^1 - \hat{p}_{k-3}^1| \leq & \nu |A(\hat{p} - p)| + |B(\hat{p} - p + \hat{q}_{k-1}^0 - q_{k-1}^0, \hat{p} + q_{k-1}^0)| \\ & + |B(p + q_{k-1}^0, p - \hat{p} + q_{k-1}^0 - \hat{q}_{k-1}^0)|. \end{aligned}$$

Using Lemma A.1 and the same techniques as in Devulder and Marion [8], one can then prove  $(A.10)_{k,1}$ . The details are left to the reader.

Next, assuming that estimate  $(A.10)_{k,i}$  has been proved up to level  $i$  ( $i \geq 1$ ), we want to prove it at level  $i + 1$ . But we have

$$\begin{aligned} p_{k-3-2i}^{i+1} - \hat{p}_{k-3-2i}^{i+1} = & \nu A(\hat{p}_{k-1-2i}^i - p_{k-1-2i}^i) \\ & + \sum_{l=0}^i \binom{i}{l} Q_m B(\delta_{k-1-2l}^l + \hat{q}_{k-1-2l} - q_{k-1-2l}, \hat{p}_{k-1-2i+2l}^{i-l} + \hat{q}_{k-1-2i+2l}^{i-l}) \\ & + \sum_{l=0}^i \binom{i}{l} Q_m B(p_{k-1-2l}^l + q_{k-1-2l}^l, \delta_{k-1-2i+2l}^{i-l} + \Delta_{k-1-2i+2l}^{i-l}), \end{aligned}$$

where  $\delta_m^l = (\hat{p}_m^l - p_m^l)$  and  $\Delta_m^l = (\hat{q}_m^l - q_m^l)$ . All the terms in the right-hand side can be estimated as in Devulder and Marion [8], by using Lemma A.1,  $(A.10)_{k,l}$  for  $1 \leq l \leq i$ , and  $(A.11)_{k-1,l}$  for  $0 \leq l \leq i$ .

(b.2) *Proof of  $(A.11)_{k,i}$ .* For  $i = [(k + 1)/2]$ , the inequality  $(A.11)_{k,i}$  is obvious, since  $q_{k-2i}^i = \hat{q}_{k-2i}^i = 0$ . Assuming now that  $(A.11)_{k,i}$  has been proved for  $j = [(k + 1)/2], \dots, i + 1$ , we can prove it at level  $i$  as in (b.1).  $\square$

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